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801 Homework 1

September 3, 2015

Problem A.11:

Let A and B be the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $C(A)^\perp$ and $C(B)^\perp$ with respect to $\mathbb{R}^4$.

Solution: The basis for the column space of A is given by the three vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then, $x \in C(A)^\perp$ must satisfy $a_i^T x = 0$ for $i = 1, 2, 3$. For $i = 1$,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

implies that $x_1 = -x_2$. For $i = 3$,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

implies that $x_4 = 0$. Thus, for $i = 2$ we have $x_3 = 0$. Thus,

$$C(A)^\perp = \left\{ \begin{bmatrix} a \\ -a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

Repeating the procedure that we did for matrix A, we see that $C(B)^\perp = C(A)^\perp$.

Problem A.14:

Find an orthogonal basis for the space spanned by the columns of

$$X = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 1 \\ 1 & 6 & 4 \end{bmatrix}.$$

Solution: To find an orthogonal basis for $C(X)$, we use Gram-Schmidt without normalizing. Let the columns of X be x_1, x_2, and x_3 so that $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$. Let $y_1 = x_1$. Then, we find y_2 to be

$$y_2 = x_2 - \frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 = x_2 - \frac{21}{6} y_1 = \begin{bmatrix} -5/2 \\ -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \\ 5/2 \end{bmatrix}.$$

Next, we see that y_3 is

$$y_3 = x_3 - \frac{\langle x_3, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 - \frac{\langle x_3, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 = x_3 - \frac{10}{6} y_1 - \frac{0}{35/2} y_2 = \begin{bmatrix} 14/6 \\ -4/6 \\ -10/6 \\ -10/6 \\ -4/6 \\ 14/6 \end{bmatrix}.$$

Calculating the dot product for y_i, y_j for $i \neq j$ yields 0. Therefore, by Gram-Schmidt, an orthogonal basis for $C(X)$ is $B = \{y_1, y_2, y_3\}$.

Problem A.16:

Let X be an $n \times p$ matrix. Prove or disprove the following statement: Every vector in $\mathbb{R}^n$ is in either $C(X)$ or $C(X)^\perp$ or both.

Solution: We will provide a counterexample to the statement. Let $n = 2$ and

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, we easily see that

$$C(X) = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$$
$$C(X)^\perp = \left\{ \begin{bmatrix} 0 \\ a \end{bmatrix} : a \in \mathbb{R} \right\}.$$

However, for the vector $(1, 1)' \in \mathbb{R}^2$, it is not in $C(X)$ nor $C(X)^\perp$.

Problem B.11:

Let A, B, and C be the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 5 & 7 \\ 1 & -5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ -3 & 0 & 1 \end{bmatrix}.$$

Use Theorem B.35: Let $o_1, ..., o_r$ be an orthonormal basis for $C(X)$, and let $O = [o_1, ..., o_r]$. Then $OO' = \sum_{i=1}^{r} o_i o_i'$ is the perpendicular projection operator onto $C(X)$., to find the perpendicular projection operator onto the column space of each matrix.

Solution: First note that $\dim(C(A)) = 2$, $\dim(C(B)) = 3$, and $\dim(C(C)) = 2$. Denote a_1, a_2, a_3 as the columns of A, b_1, b_2, b_3 as the columns of B, and c_1, c_2, c_3 as the columns of C. We first need to find an orthonormal basis $\mathcal{B}_A, \mathcal{B}_B$, and $\mathcal{B}_C$ for $C(A), C(B)$, and $C(C)$, respectively; use Gram-schmidt to do this. By Gram-Schmidt, we let the first element of each basis be

$$ba_1 = \frac{a_1}{||a_1||} = \frac{1}{\sqrt{6}} a_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$bb_1 = \frac{b_1}{||b_1||} = b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$bc_1 = \frac{c_1}{||c_1||} = \frac{1}{\sqrt{14}} c_1 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}.$$

We calculate the following vectors

$$ua_2 = a_2 - \frac{\langle a_2, a_1 \rangle}{\langle a_1, a_1 \rangle} a_1 = a_2 - \frac{0}{6} a_1 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$$

$$ub_2 = b_2 - \frac{\langle b_2, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 = b_2 - \frac{0}{1} b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$uc_2 = c_2 - \frac{\langle c_2, c_1 \rangle}{\langle c_1, c_1 \rangle} c_1 = c_2 - \frac{14}{14} c_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Then, normalizing these, we have the second element of each basis to be

$$ba_2 = \frac{ua_2}{||ua_2||} = \frac{1}{\sqrt{50}} ua_2 = \begin{bmatrix} 0 \\ 5/\sqrt{50} \\ -5/\sqrt{50} \end{bmatrix}$$

$$bb_2 = \frac{ub_2}{||ub_2||} = \frac{1}{1} ub_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$bc_2 = \frac{uc_2}{||uc_2||} = \frac{1}{\sqrt{27}} uc_2 = \begin{bmatrix} 3/\sqrt{27} \\ 3/\sqrt{27} \\ 3/\sqrt{27} \end{bmatrix}.$$

We calculate the following vectors

$$ua_3 = a_3 - \frac{\langle a_3, a_1 \rangle}{\langle a_1, a_1 \rangle} a_1 - \frac{\langle a_3, ua_2 \rangle}{\langle ua_2, ua_2 \rangle} ua_2 = a_3 - \frac{12}{6} a_1 - \frac{50}{50} ua_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$ub_3 = b_3 - \frac{\langle b_3, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 - \frac{\langle b_3, ub_2 \rangle}{\langle ub_2, ub_2 \rangle} ub_2 = b_3 - \frac{0}{1} b_1 - \frac{0}{1} ub_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$uc_3 = c_3 - \frac{\langle c_3, c_1 \rangle}{\langle c_1, c_1 \rangle} c_1 - \frac{\langle c_3, uc_2 \rangle}{\langle uc_2, uc_2 \rangle} uc_2 = c_3 - \frac{0}{14} c_1 - \frac{9}{27} uc_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that ua_3 and uc_3 are the zero vector, which makes sense because the dimensions of $C(A)$ and $C(C)$ are both 2. Lastly, since $||ub_3|| = 1$, we have $b_3 = ub_3$. Therefore, we have

$$\mathcal{B}_A = \left\{ \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ 5/\sqrt{50} \\ -5/\sqrt{50} \end{bmatrix} \right\}$$

$$\mathcal{B}_B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{B}_C = \left\{ \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{27} \\ 3/\sqrt{27} \\ 3/\sqrt{27} \end{bmatrix} \right\}$$

are orthonormal bases for $C(A), C(B)$, and $C(C)$, respectively. Now let

$$O_A = \begin{bmatrix} 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & 5/\sqrt{50} \\ 1/\sqrt{6} & -5/\sqrt{50} \end{bmatrix}, \quad O_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad O_C = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{27} \\ 2/\sqrt{14} & 3/\sqrt{27} \\ -3/\sqrt{14} & 3/\sqrt{27} \end{bmatrix},$$

to find the perpendicular projection operators

$$O_A O_A' = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}, O_B O_B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, O_C O_C' = \begin{bmatrix} 17/42 & 10/21 & 5/42 \\ 10/21 & 1/2 & -2/21 \\ 5/42 & -2/21 & 41/42 \end{bmatrix}$$

onto the column spaces of $C(A), C(B)$, and $C(C)$, respectively, by Theorem B.35.

Problem B.12:

Show that for a perpendicular projection matrix M,

$$\sum_i \sum_j m_{ij}^2 = r(M).$$

Solution: Assume that M is an $n \times n$ matrix. Note that since M is a perpendicular projection matrix, it is idempotent and $M = M'M = MM'$ by problem $B.13$. Because M is idempotent, $r(M) = \text{tr}(M) = \text{tr}(MM')$. Also, by definition of trace,

$$\begin{aligned} r(M) = \text{tr}(MM') &= \sum_{i=1}^{n} (MM')_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} m'_{ji} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} m_{ij} = \sum_i \sum_j m_{ij}^2. \end{aligned}$$

This proves the equality.

Problem B.17:

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

(a) Show that A is a projection matrix.

Solution: Note that

$$C(A) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}.$$

Also, for any $x = (x_1, x_2)'$, we have $Ax = (x_2, x_2)' \in C(A)$. Therefore, A is a projection matrix onto $C(A)$.

(b) Is A a perpendicular projection matrix? Why or why not?

Solution: Note that for A to be a perpendicular projection matrix, it must be symmetric by Theorem B.33. However, $A \neq A'$ and therefore is not symmetric. So, A is not a perpendicular projection matrix.

(c) Describe the space that A projects onto and the space that A projects along. Sketch these spaces.

Solution: The matrix A projects onto $C(A)$ which was found in part (a). Thus, we see that A projects onto the line $y = x$. Also, A projects along the space $N(A)$, which is given by

$$N(A) = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

Therefore, A projects along the x-axis.

(d) Find another projection operator onto the space that A projects onto.

Solution: Another simple projection matrix onto $C(A)$ would be the matrix

$$B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Therefore, for $x \in C(A)$, $Bx = x$.

Problem B.10:

Show that the matrix B given below is positive definite, and find a matrix Q such that $B = QQ'$. (Hint: The first row of Q can be taken as $(1, -1, 0)$.)

$$B = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Solution: First we show that B is positive definite by appealing tot he definition. So,

$$\begin{aligned}
x'Bx &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -x_1 + x_2 \\ x_1 + 2x_3 \end{bmatrix} \\
&= 2x_1^2 - x_1x_2 + x_1x_3 - x_1x_2 + x_2^2 + x_1x_3 + 2x_3^2 \\
&= 2x_1^2 - 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_3^2 \\
&= x_1^2 + (x_1 - x_2)^2 + 2x_1x_3 + 2x_3^2 \\
&= (x_1 + x_3)^2 + (x_1 - x_2)^2 + x_3^2 \\
&> 0.
\end{aligned}$$

Therefore, B is positive definite. To find Q, we solve the system

$$B = \begin{bmatrix} 1 & -1 & 0 \\ x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_4 \\ -1 & x_2 & x_5 \\ 0 & x_3 & x_6 \end{bmatrix}.$$

This implies the following equations

$$\begin{aligned}
x_1 - x_2 &= -1 \\
x_4 - x_5 &= 1 \\
x_1^2 + x_2^2 + x_3^2 &= 1 \\
x_1x_4 + x_2x_5 + x_3x_6 &= 1 \\
x_4^2 + x_5^2 + x_6^2 &= 2.
\end{aligned}$$

We have 6 unknowns, but only 5 equations. Let $x_5 = 0$. Then $x_4 = 1$. Now we're left with

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_1^2 + x_2^2 + x_3^2 &= 1 \\ x_1 + x_3 x_6 &= 1 \\ 1 + x_6^2 &= 2. \end{aligned}$$

So, we have that $x_6 = \pm 1$. Take x_6 to be 1. Lastly, we solve

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_1^2 + x_2^2 + x_3^2 &= 1 \\ x_1 + x_3 &= 1. \end{aligned}$$

Take $x_1 = 0$ and so $x_2 = 1$, which gives that $x_3 = 1$. Therefore, Q is the matrix

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Multiplying QQ' gives that $B = QQ'$. Therefore, we have found Q.

Problem B.13:

Prove that if $M = M'M$, then $M = M'$ and $M = M^2$.

Solution: Let $M = M'M$. Then, we have $M' = (M'M)' = M'M = M$. It follows that $M = M'M = MM = M^2$.